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Abstract

Halfline tests studied in this paper are t type tests of inequality constraints under the alternative hypothesis. We first study the halfline tests which are the most stringent and somewhere most powerful tests. We also propose a halfline test that has a computational advantage. Some theoretical properties concerning a class of halfline tests are presented. Simulation studies are carried out to examine the performance of halfline tests against some existing tests. Results of our simulation studies reveal that halfline tests can outperform other tests and possess a robustness property against the normality assumption.

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1 Introduction

This paper is concerned with hypothesis tests involving inequality constraints under alternatives. We study halfline tests which are a type of t tests. Such tests have been studied previously by some authors (e.g., Schaafsma and Smid (1966), Hillier (1986), King and Smith (1986) and Akharif and Hallin (2003)).

For ease of exposition we present our findings concerning some theoretical aspects of halfline tests in the framework of testing the multivariate mean. Let $X_i = (x_{1i}, ..., x_{pi})'$, $i = 1, ..., n$, be the independent p-dimensional random vector, each from a distribution with the common mean $\mu = (\mu_1, ..., \mu_p)'$. Also let $\bar{X} = (\bar{x}_1, ..., \bar{x}_p)'$ be the sample mean. The hypotheses of interest are

$$H_0 : \mu = 0 \qquad vs \qquad H_1 : \mu \geq 0, \tag{1}$$

where the equality and inequality are to be taken on a coordinate-wise basis with strict inequality holding for at least one coordinate under H_1. Note that if a scalar is involved, then each coordinate of the vector is to be compared with the scalar; this remains the case in this paper unless otherwise stated. Throughout the paper Σ is assumed to be positive definite.

The Likelihood Ratio (LR) based approach to the testing problem (1) has been most common in the literature (e.g., Kudô (1963), Gouriéroux et al. (1982), Wolak (1987), Wolak (1989) and Maasoumi (2001)). However, there are some limitations with LR-based approaches. In the case of normal distribution with known Σ, the LR test statistic is shown to follow the null distribution of a mixture of central Chi-square random variables in which the weighting probability depends on Σ (Kudô (1963)). When p is large, computing the weighting probability can be challenging; numerical methods are even needed when p is as small as 4 (Wolak (1987)). When Σ is unknown, the exact null distribution of LR tests is unknown. Perlman (1969) derived a probability bound for computing the critical region under the normality assumption. Perlman's bound has been adopted by Silvapulle and Silvapulle (1995) for testing ARCH effects and by Rosen (2008) for inference in moment equality/inequality models. However, Perlman's bound can be conservative. In fact, it led to the discovery of a uniformly more powerful test by Tang (1994).

An alternative approach to the testing problem (1) is halfline tests, which were suggested around the same time that LR tests were proposed (see discussion in Bartholomew (1961)). Halfline tests use a single halfline in the restricted parameter space to represent the entire restricted parameter space. The sample mean is then projected onto the halfline to form a test statistic. The resultant test statistic is a t-ratio and is most powerful against the alternatives lying along the halfline when Σ is known or known up to a multiplier under the normal distribution. The drawback of halfline tests is that, as alternatives move away from the halfline, power loss (shortcoming) increases. It is of interest to see the effect of power loss in comparison with other tests such as LR tests. Because of the nature of t tests one expects that halfline tests possess a robustness property against the normality assumption. Delaigle et al. (2011) demonstrated such a robustness property with the aid of the bootstrap procedure for estimating Σ in very high dimension problems such as sparse signal detection.

The idea of halfline tests was first suggested by Abelson and Tukey (1963) in the form of contrast tests. Schaafsma and Smid (1966) (SS hereafter) extended the work of Abelson and Tukey (1963) and proposed the most stringent somewhere most powerful (MSSMP) tests, in which the halfline is chosen to minimize the maximum angle between the halfline and the edges of the polyhedral cone resulting from the transformation of the positive orthant that is the restricted parameter space under H_1 (which is discussed in more detail in Section 2). Under the assumption of normal distributions with the known covariance Σ, SS studied a form of MSSMP tests where the halfline forms the same angle with the edges of the polyhedral cone, hence we call it equiangular MSSMP tests (EAMSSMP) in this paper. In cases where the EAMSSMP test does not exist, SS outlined a method for conducting MSSMP tests, but did not discuss the implementation of the method. Hillier (1986) proposed a procedure for finding the halfline in the cases of $p = 2, 3$ such that the minimax angle is achieved. To the best of our knowledge, there are no studies in the literature for implementing MSSMP tests for the cases of $p > 3$. King and Smith (1986) studied one-sided t tests in linear regression models. They demonstrated uniformly most powerful invariant property in their tests against certain alternatives. However, they did not consider the minimax angle objective. Recently, there appears to have been renewed interest in MSSMP tests. Leuraud and Benichou (2001), Leuraud and Benichou (2004) and Leuraud and Benichou (2006) extended SS's EAMSSMP tests for testing a monotonic trend with categorical data. They found that EAMSSMP tests perform favorably over other existing tests. Akharif and Hallin (2003) extended EAMSSMP tests for testing the random coefficient effect in autoregressive modes.

In this paper we first establish a necessary and sufficient condition for the admissibility of SS's EAMSSMP tests under the assumption of normality with known Σ. When the condition is met, we provide a simple closed-form expression for the EAMSSMP test statistic, which is shown to be invariant to the transformation matrix involved. However, as suggested by our condition, EAMSSMP tests can easily fail to exist. We then propose an algorithm for implementing MSSMP tests by modifying the method suggested by SS, which we show cannot guarantee an MSSMP test. The proposed algorithm can be easily implemented, but may be potentially computationally demanding if p is very large. A halfline test is then proposed for computational simplicity. The proposed test does not warrant the most stringent (MS) property, but retains the somewhere most powerful (SMP) property. We show that MSSMP tests including EAMSSMP tests are unbiased and consistent. We also show that the proposed halfline test enjoys such properties. However, such unbiasedness and consistency properties may not be shared by halfline tests, in which the halfline is held fixed with respect to Σ as suggested in the literature by, for example, King and Smith (1986) and Akharif and Hallin (2003). In the cases of unknown Σ under normal or non-normal distributions, with the use of the unbiased estimate instead, consistency is shown to remain while unbiasedness holds asymptotically under some further conditions. Simulation studies are carried out to compare the performance of halfline tests against some tests proposed in the literature. The results of our simulation studies suggest that, in the case of the normal distribution with known Σ, halfline tests have better finite sample powers compared to LR tests when μ lies in the areas surrounding the halfline while they are inferior to LR tests when μ lies on the boundary of the restricted parameter space. In the case of unknown Σ, halfline tests can also have a better finite sample performance on the boundary. Furthermore, our simulation results show that halfline tests are more robust than LR-based tests against the normality assumption. In fact, they can even perform better over the distribution-free test proposed by Larocque and Labarre (2004) in some non-normal distribution cases.

The remainder of the paper is organized as follows. preliminary material is presented in Section 2. Section 3 studies EAMSSMP tests. A procedure is proposed in Section 4 for implementing MSSMP tests. Section 5 proposes a halfline test that has a computational advantage. In Section 6 we study some properties concerning halfline tests. Simulation studies are carried out in Section 7. Some concluding remarks are provided in Section 8. The proofs of some lemmas/theorems are relegated to the appendix while other proofs are presented where the corresponding lemmas/theorems appear in the main body of the paper.

2 Preliminaries

Suppose that the distribution is normal and Σ is known. Let $A = (a_1, ..., a_p)$, where a_j, $j \in \mathbf{p} = \{1, ..., p\}$, is a $p \times 1$ vector, such that $A'A = \Sigma^{-1}$. Let l_a be the halfline emanating from the origin through the point defined by a. Denote C_A the closed pointed polyhedral cone (closed convex cone with the vertex at the origin) with the edges defined by l_{a_j}, $\forall j$. Denote H^+ the half linear space such that $H_A^+ = \{v : u'v \geq 0\}$, where $u \in C_A$ is a fixed vector. The pointed polyhedral cone implies that $C_A \subset H_A^+$.

If we apply a transformation $Z = A\bar{X}$, then $\sqrt{n}Z \sim N(\eta, I)$, where $\eta = A\mu$ and I is the identity matrix. Under the linear transformation A, the restricted parameter space defined by H_1, which is the positive orthant, C_I, is transformed to C_A. The test

$$T = \sqrt{n}b'Z, \tag{2}$$

where $b \in C_A$ is a $p \times 1$ non-null fixed vector, is uniformly most powerful against the alternatives $\mu \in \{\mu : A\mu = \epsilon b\}$, where ϵ is a positive scalar, or $\eta \in l_b$; that is all μ such that $\eta = A\mu$ lying on the halfline l_b. (Note that for brevity of presentation, we have assumed that the cone C and halfline l do not include the origin here and thereafter.) However, T is not most powerful for other alternatives. Thus, halfline tests for hypotheses (1) are only somewhere most powerful.

T can be equivalently rewritten as

$$T = \sqrt{n} b_I' \Sigma^{-1} \bar{X}, \tag{3}$$

where $b_I = A^{-1}b \in C_I$. Obviously T is invariant to the scaling of b, so we let b be a unit vector; $\|b\| = 1$, where $\|\cdot\|$ denotes the Euclidean norm. Then, under H_0, $T \sim N(0,1)$ and we reject H_0 for a large T.

Denote $\varphi(l_b, l_\eta)$ the angle between the halflines l_b and l_η. For $\eta \in C_A$, it can be shown that the power of the test T is

$$\beta(\varphi(l_b, l_\eta)) = 1 - \Phi(c_\alpha - \sqrt{n} r \cos\varphi(l_b, l_\eta)), \tag{4}$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal random variable, c_α the critical value at the α significance level, $r = \|\eta\|$ and

$$\cos\varphi(l_b, l_\eta) = r^{-1} \langle \eta, b \rangle, \tag{5}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product. Clearly $\beta(\varphi)$ is a strictly decreasing function of φ within the H_A^+, i.e., $0 \leq \varphi \leq \pi$. Furthermore, if $0 \leq \varphi \leq \pi/2$, $\beta(\varphi) \geq \alpha$ with the equality holds at $\varphi = \pi/2$ and $\beta(\varphi) < \alpha$ if $\pi/2 < \varphi \leq \pi$.

Let β^* be the power envelope for a fixed $r > 0$, which occurs at $\varphi = 0$ or $b = r^{-1}\eta$. Because η is unknown, one may undesirably choose a $b \neq r^{-1}\eta$, which results in $\varphi \neq 0$. The shortcoming of the test is

$$\triangle\beta(\varphi) = \beta^* - \beta(\varphi). \tag{6}$$

SS suggested choosing b so that the maximum shortcoming of the test for $l_b, l_\eta \subset C_A$

is minimized, i.e.,

$$b = \arg \inf_{l_b \subset C_A} \sup_{l_\eta \subset C_A} \triangle\beta(\varphi(l_b, l_\eta)).$$

Clearly this is equivalent to the minimization of the maximum angle between the halfline l_b and the edges of C_A. The resultant test is most stringent over a class of halfline tests $\{T(b) : b \in R^p\}$. It is also most powerful against the alternatives lying along the halfline. SS showed that an EAMSSMP test exists if $b \in C_A$ is found such that the angles $\varphi_j = \varphi(l_b, l_{a_j})$, $\forall j$, are all equal; that is l_b is the equiangular halfline to the edges of C_A, l_{a_j}, $\forall j$.

3 Admissibility of EAMSSMP tests

We study EAMSSMP tests in this section. We first establish a necessary and sufficient condition for the admissibility of EAMSSMP tests; we show that with the violation of the condition EAMSSMP tests fail to be most stringent and are not most powerful against any alternatives, hence EAMSSMP tests are not admissible.

Condition 1. *The covariance matrix* Σ *satisfies* $\Sigma s \geq 0$*, where* $s = (\sqrt{\Sigma_{11}^{-1}}, .., \sqrt{\Sigma_{pp}^{-1}})'$ *and* Σ_{jj}^{-1} *is the* jj*th element of* Σ^{-1}*.*

Lemma 1. *Condition 1 is the necessary and sufficient condition for the admissibility of EAMSSMP tests.*

Proof. For a halfline l_b with $\|b\| = 1$, $\cos\varphi_j(l_b, l_{a_j}) = a_j'b/\|a_j\|$. If l_b forms the same angle with l_{a_j}, $\forall j$, we have $\cos\varphi_j(l_b, l_{a_j}) = \delta$, $\forall j$, which is equivalent to $A'b = \delta s$. Hence $b = \delta A'^{-1}s$. Because $l_b \subset C_A$ (or $b \in C_A$) is equivalent to $A^{-1}b \in C_I$, we further have $A^{-1}b = \delta\Sigma s \geq 0$. Because $C_A \subset H_A^+$, we have $0 < \varphi_j < \pi/2$, $\forall j$, hence, $0 < \delta < 1$. Therefore, $\Sigma s \geq 0$. In the sequel the test $T = \sqrt{n}b'Z$ is the EAMSSMP test. For necessity, suppose that the EAMSSMP test exists with some elements of

Σs being negative, then $b \notin C_A$, which means the test is nowhere most powerful for alternatives in C_A. Furthermore, projecting such $l_b \notin C_A$ reduces the maximum shortcoming, so the test is not most stringent. In summary, the test using $b \notin C_A$ is not MSSMP, which violates the assumption. ■

Note that a choice of A is not unique, for example, A is chosen to be the triangular matrix such that the elements below the diagonal are 0, or A is computed as $A = D^{-1/2}V'$, where D is the diagonal matrix with the eigenvalues of Σ being on the diagonal and V is the corresponding eigenvectors in columns, or A is computed according to the methods presented in Tang et al. (1989). However, a different choice of A does not affect the EAMSSMP test, as stated in the following lemma.

Lemma 2. *Under Condition 1, the EAMSSMP test of* (2) *(or* (3)*) is unique and invariant to a choice of A, and can be written as*

$$T_x = \sqrt{n}\delta s'\bar{X}. \tag{7}$$

where $\delta = (s'\Sigma s)^{-1/2}$.

Proof. As shown in the proof of Lemma 1, if the EAMSSMP test exists, then $b = \delta A'^{-1}s$ and $b'b = 1$. Hence $\delta = (s'\Sigma s)^{-1/2}$. From Equation (2), we have $T = \sqrt{n}\delta s'A^{-1}A\bar{X} = \sqrt{n}\delta s'\bar{X}$, which is unique and does not depend on A. ■

Because EAMSSMP tests are invariant to A, alternative to Equation (7), one may compute the test statistic as follows. Compute $b = b_a/\|b_a\|$, where $b_a = A'^{-1}s$, and $T_a = \sqrt{n}b'A\bar{X}$.

SS (p. 1167) argued that EAMSSMP tests exist in a great number of applications. However, we find that Condition 1 can fail easily. This is confirmed in our computation exercises. For example, let $\Sigma = (U'U)^{-1}$ and U is generated from $U \sim N(0, \Sigma_U)$

with $n = 20$ and the element at the position (j, j_1), $j, j_1 \in \mathbf{p}$, of Σ_U being $0.5^{|j-j_1|}$. (Note that there exists the equiangular halfline for this structure of Σ_U.) In a computational exercise of 100 replications there are 53 for $p = 5$ and 99 for $p = 10$, in which one or more elements of Σs is less than $-\varepsilon$, where ε is a small positive value, say -10^{-10} for preventing numerical error, hence EAMSSMP tests fail to exist. In fact, the angle between the equiangular halfline and the halfline used for implementing MSSMP tests presented in the next section can be large. This means that EAMSSMP tests can be considerably improved by MSSMP tests presented in the next section in terms of minimizing the maximum shortcoming if Condition 1 fails to be satisfied. We now show analytically how easily the failure of Condition 1 can occur in the case of $p = 3$. Suppose that Σ is the correlation matrix with ρ_{jj_1}, $j \neq j_1$, being the correlation coefficients. $\Sigma^{-1}_{jj} = B_{jj}/\left|\Sigma\right|$, where B_{jj} is the cofactor of the jjth element of Σ. Then the jth element of Σs is $\left|\Sigma\right|^{-1/2} \sum_{j_1=1}^{3} \rho_{jj_1} \sqrt{B_{jj}}$, where $\rho_{jj} = 1$. It is easy to find some values of ρ_{jj_1} such that some element of Σs is negative. For example, if $\rho_{12} = \rho_{13} = -0.5$ and $\rho_{23} \geq 0.5$, then the first element of Σs is negative.

However, when $p = 2$, Condition 1 is always satisfied as shown in the proof of the following lemma.

Lemma 3. *In the case of $p = 2$, there always exists the EAMSSMP test.*

Proof. Suppose

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}, \tag{8}$$

where $\sigma_1 > 0$, $\sigma_2 > 0$ and $-1 < \sigma_{12}/(\sigma_1\sigma_2) < 1$. We have

$$s = \frac{1}{\sqrt{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}} \binom{\sigma_2}{\sigma 1},$$

so

$$\Sigma s = \frac{\sigma_1\sigma_2 + \sigma_{12}}{\sqrt{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} > 0.$$

Hence by Lemma 2 the test

$$T_x = \sqrt{\frac{n}{2\sigma_1\sigma_2(\sigma_1\sigma_2 + \sigma_{12})}} (\sigma_2 \quad \sigma_1)\bar{X} \tag{9}$$

is of EAMSSMP. ■

4 Implementation of MSSMP tests

When the EAMSSMP test does not exist, SS (p. 1168) suggested finding b_k such that l_{b_k} is the equiangular halfline in C_{A_k} which is a k-dimensional face of C_A satisfying the conditions (i) $l_{b_k} \subset C_{A_k}$ and (ii) $\varphi(l_{b_k}, l_{a_{j_1}}) > \varphi(l_{b_k}, l_{a_{j_2}})$, $l_{a_{j_1}} \subset C_{A_k}$, $l_{a_{j_2}} \subset C_A \backslash C_{A_k}$. However, because it is possible that $C_A \subset C_{A_k}$ for a given C_A, it is not guaranteed that $l_{b_k} \subset C_{A_k}$ leads to $l_{b_k} \subset C_A$. Therefore, it is possible that there exists a b_k that satisfies the conditions (i) and (ii), but fails to satisfy $l_{b_k} \subset C_A$. (Note that one could easily modify the algorithm discussed below to find such b_k. The Matlab program for implementing the algorithm is available from the author on request.) In the sequel, the test using such b_k is nowhere most powerful for the restricted alternatives, nor most stringent because the maximum shortcoming can further be minimized as the halfline is moving towards C_A. We suggest that the above condition (i) is replaced with (i') $l_{b_k} \subset C_A$.

We now discuss an algorithm for implementing MSSMP tests. Denote $\tilde{A}_k = (\tilde{a}_1, ..., \tilde{a}_k)$, where $\tilde{a}_j = a_j / \|a_j\|$, $k < p$. By Equation (2.6) of Efron et al. (2004) the equiangular halfline l_{b_k} that forms the same angle with each edge of the polyhedral cone $C_{\tilde{A}_k}$ is computed as

$$b_k = \delta_k \tilde{A}_k G_k^{-1} 1_k, \tag{10}$$

where $\delta_k = (1_k' G_k^{-1} 1_k)^{-1/2}$, $G_k = \tilde{A}_k' \tilde{A}_k$ and 1_k is an $k \times 1$ vector of 1. Note that the positivity property of the cone C_A means $C_{\tilde{A}_k} = C_{A_k}$. As shown in Efron et al. (2004) $\|b_k\| = 1$ and $\cos\varphi(l_{b_k}, l_{\tilde{a}_j}) = \delta_k \leq 1$, $\forall \tilde{a}_j \in C_{\tilde{A}_k}$.

Denote the set $\mathbf{k} = \{\tilde{j} = 1, ..., k : \tilde{j} \in \mathbf{p}, k < p\}$ and $\mathbf{k}^c = \mathbf{p} \backslash \mathbf{k}$. If $\mathbf{k}_1 \subseteq \mathbf{k}_2$, then by Lemma 5 in Efron et al. (2004) $\delta_{k_1} \geq \delta_{k_2}$ or equivalently $\varphi(l_{b_{k_1}}, l_{\tilde{a}_{\tilde{j}_1}}) < \varphi(l_{b_{k_2}}, l_{\tilde{a}_{j_2}})$, $\tilde{j}_1 \in \mathbf{k}_1$, $\tilde{j}_2 \in \mathbf{k}_2$. This implies that the smaller k is, the more likely that $\varphi(l_{b_k}, l_{\tilde{a}_{\tilde{j}_1}}) \leq \varphi(l_{b_k}, l_{\tilde{a}_{j_2}})$, $\tilde{j}_1 \in \mathbf{k}$, $\tilde{j}_2 \in \mathbf{k}^c$. Therefore, we suggest that the algorithm starts at $k = p - 1$. Given an $\tilde{A}_k$, compute b_k according to Equation (10). If

$$\tilde{A}^{-1} b_k \geq 0, \quad \tilde{A}_{k^c}' b_k > \delta_k, \tag{11}$$

where $\tilde{A} = \tilde{A}_p$, $\tilde{A}_{k^c} = \{\tilde{a}_{\tilde{j}_2} : \forall \tilde{j}_2 \in \mathbf{k}^c\}$, stop the procedure and replace b with b_k in (2) or replace b_I with $A^{-1} b_k$ in (3) to carry out the test. Otherwise, continue at a new $\tilde{A}_k$ (note that there is a total $p!/[(p-k)!k!]$ such $\tilde{A}_k$), or move to the next $k - 1$ if there is no new $\tilde{A}_k$ available, until such b_k is found.

The next lemma provides justifications on the termination of the procedure once b_k that satisfies the inequalities (11) is found. The earlier termination of the procedure can result in a considerable saving of computations.

Lemma 4. *If Condition 1 is not met, there exists a unique* b_k, $2 \leq k \leq p - 1$ *such that the inequalities* (11) *are satisfied.*

The next lemma shows that the invariance property of EAMSSMP tests is shared by MSSMP tests.

Lemma 5. *The MSSMP test described above is invariant to A.*

Proof. Let S be the diagonal matrix with s on the diagonal; $S = diag(s)$. Because

$$\tilde{A}'\tilde{A} = \begin{pmatrix} \tilde{A}'_k\tilde{A}_k & \tilde{A}'_k\tilde{A}_{k^c} \\ \tilde{A}'_{k^c}\tilde{A}_k & \tilde{A}'_{k^c}\tilde{A}_{k^c} \end{pmatrix} = S^{-1}\Sigma^{-1}S^{-1},$$

$\tilde{A}'_k\tilde{A}_k$, $\tilde{A}'_k\tilde{A}_{k^c}$ and $\tilde{A}'_{k^c}\tilde{A}_{k^c}$, $1 < k < p$ are invariant to A. Hence, G_k^{-1} is also invariant to A. The MSSMP test statistic is

$$T = \sqrt{n}(1'_kG_k^{-1}1_k)^{-1/2}1'_kG_k^{-1}\tilde{A}'_kA\bar{X}, \tag{12}$$

which is clearly invariant to A. Similarly, it can be easily shown that the inequalities (11) are invariant to A. Therefore, MSSMP tests are invariant to A. ∎

In the case of $p = 3$, when the equiangular halfline l_{b_p} falls outside C_A, the closest face of C_A to l_{b_p} is the one that has the largest angle formed by its two edges (Hillier (1986)). The algorithm described above implies that the MSSMP test arises from the projection of l_{b_p} onto C_A; the projection onto the closest face of C_A, so that the projection is the halfline that forms the same angle with the two edges on the closest face and forms a smaller angle with the remaining edge. Therefore, the algorithm may be simplified to Hillier's procedure as follows. Compute δ_k for $k \in \mathbf{k} = \{j, j+1 : j \in \mathbf{p}\}$, where $j+1$ is set to 1 if $j+1 = 4$, then compute the test statistic (12) using k which corresponds to the smallest δ_k.

5 Proposed halfline tests

The algorithm discussed in the previous section consumes little time for an implementation of MSSMP tests when p is small or moderate, say up to 20. Therefore, it is effective for most applications. This suggests a computational advantage of MSSMP tests over many other tests proposed in the literature. However, when p is very large, an implementation of MSSMP tests can be computationally demanding. For example, for $p = 100$, if the optimal b_k does not exist at $k = p-1$, $p-2$. The computation can quickly expand for searching for b_k at $k = p-3$. This section proposes a halfline test, which has a better computational advantage. The proposed test does not warrant the MS property, but retains the SMP property.

Suppose that the true η lies on the jth edge of C_A. By Equation (6) the maximum shortcoming of the test T is $\triangle\beta_j(\varphi(l_b, l_{a_j})) = \beta^* - \beta(\varphi(l_b, l_{a_j}))$. One may wish to find b so that the sum of the maximum shortcoming for η possibly lying on l_{a_j}, $j \in \mathbf{p}$, $\sum_{j=1}^{p} \triangle\beta_j(\varphi(l_b, l_{a_j}))$ is minimized. From Equation (4), this is equivalent to minimizing $\sum_j \Phi(c_\alpha - \sqrt{n}r\cos\varphi(l_b, l_{a_j}))$. Because the evaluation of $\Phi(\cdot)$ involves numerical integration, for ease of computation one may approximate $\Phi(\cdot)$ by the Taylor expansion at $\varphi_j = 0$ for a given n as

$$\Phi(c_\alpha - \sqrt{n}r\cos\varphi_j) = \Phi(w) - \sqrt{n}r\phi(u)(\cos\varphi_j - 1),$$

where $u = c_\alpha - \sqrt{n}r$. Therefore, instead of minimizing $\sum_j \triangle\beta_j$ we maximize

$$\sum_j \cos\varphi_j(l_b, l_{a_j}) = 1_p'\tilde{A}'b,$$

The maximization is subject to the constraints discussed below:

$$\max_b \ 1_p'\tilde{A}'b \quad s.t. \quad \tilde{A}^{-1}b \geq 0, 0 < \tilde{A}'b \leq \delta_l \text{ and } \|b\| = 1, \tag{13}$$

where δ_l is the largest among $\{\delta_k : \mathbf{k} = \{j, j+1\}, j, j+1 \in \mathbf{p}\}$; the δ_k values computed for each of p two-dimensional faces of C_A. The first constraint, $\tilde{A}^{-1}b \geq 0$, is to ensure the optimum $b \in C_A$. The second constraint, $\tilde{A}'b > 0$, is to ensure $\varphi_j < \pi/2$. The third constraint, $\tilde{A}'b < \delta_l$, is to ensure $\varphi_j > \varphi'$, where φ' is the smallest angle among those between the edge and the equiangular halfline on each of the p two-dimensional faces of C_A. This constraint is to prevent l_b from overlapping with any edges of C_A for $p \geq 2$. If the constraint is set to the equality for $p = 2$, then it becomes the EAMSSMP test. The last constraint, $\|b\| = 1$, is the unit length constraint.

Lemma 6. *Given A, there exists a unique solution for the constrained optimization problem stated in* (13).

The above optimization problem can be easily solved using, for example, the procedure FMINCON in Matlab. Note that one may also use the Matlab procedure LINPROG by changing the last constraint to $b \leq 1$ and the resultant test also enjoys the theoretic properties studied in the next section. We wish to point out that, unfortunately, the solution of the optimization problem stated in (13) is not invariant to A. However, the constraints imposed dampen the effect of a different choice of A. Nevertheless, MSSMP tests should be preferred over the above proposed halfline tests unless p is very large.

6 Properties

In this section we show that MSSMP tests, including EAMSSMP tests and our proposed halfline tests, are unbiased and consistent when Σ is known or known up to a multiplier. The latter case arises in tests of regression coefficients in linear regression models (Hillier (1986) and King and Smith (1986)) for example. In the case that Σ is unknown, unbiasedness holds asymptotically while consistency remains. However, we show that some halfline tests suggested in the literature do not enjoy such properties.

We first establish a necessary and sufficient condition for unbiasedness and consistency of halfline tests when Σ is known under normal distributions.

Condition 2. *The covariance of normal distribution satisfies* $\Sigma^{-1}b_I > 0$.

Theorem 1. *Condition 2 is the necessary and sufficient condition for the halfline tests represented by Equations* (2) *or* (3) *to be unbiased and consistent.*

The consistency and unbiasedness of EAMSSMP, MSSMP and our proposed halfline tests immediately follow the above theorem as stated in the following corollary.

Corollary 1. *The proposed halfline test and MSSMP tests, including EAMSSMP tests, are unbiased and consistent when the covariance of normal distribution* Σ *is known.*

In the case where Σ is known up to a multiple, i.e., $\Sigma = \sigma^2 Q$, where Q is known, let $\hat{\sigma}^2$ be the unbiased estimator of σ^2 having the Chi-square distribution with the n' degrees of freedom and being independent of $\bar{X}$.

Corollary 2. *The halfline test statistics* T *and* T_x *in Equations* (2), (3) *and* (7) *are modified by* $\hat{\sigma}^{-1}T$ *and* $\hat{\sigma}^{-1}T_x$ *with the replacement of* Σ *with* Q. *The modified test is unbiased and consistent and have the null distribution of the student* t *distribution with* n' *degrees of freedom.*

Authors such as King and Smith (1986) and Akharif and Hallin (2003) (p. 697)) have suggested that b_I may be chosen to be $b_I = (1, ..., 1)'$, which says that l_{b_I} is the equiangular halfline of C_I. We now show that such a choice does not guarantee unbiasedness and consistency. In fact, this is true for any fixed b_I with respect to Σ.

Theorem 2. *Halfline tests with* b_I *held fixed with respect to* Σ *are biased and inconsistent.*

In the case where Σ is unknown and is replaced with its unbiased estimate

$$\hat{\Sigma} = (n-1)^{-1} \sum_{i=1}^{n} (X_i - \mu)(X_i - \mu)'$$

under the normal and non-normal distributions, the next Lemma shows that the above results hold asymptotically.

Condition 3. *There exist matrices B_n such that $B_n \sum_{i=1}^{n} X_i - n\mu \xrightarrow{p} N(0, I)$ as $n \to \infty$, which is said that X is in the generalized domain of attraction.*

Lemma 7. *Under Conditions 2 and 3, with the replacement of Σ with $\hat{\Sigma}$, halfline tests are asymptotically unbiased and consistent.*

7 Simulation studies

In this section we report simulation studies that examine the performance of MSSMP halfline tests, including the fixed halfline tests (called the FT test hereafter), against LR-based tests. To gauge the robustness of halfline tests we also perform MSSMP halfline tests against the distribution-free sign test that was proposed by Larocque and Labarre (2004) for testing the hypotheses (1), henceforth the LLS test. We study the cases of normal distributions with known and unknown Σ and non-normal distributions with unknown Σ. The halfline test T considered is the EAMSSMP test and the FT test with the test statistic computed by Equation (3) with $b_I = (1 \quad 1)'$. For the case of the normal distribution with known Σ, we compare it with the LR test of Kudô (1963), with the test statistic computed as

$$LRK = n\bar{X}'\Sigma^{-1}\bar{X} - f, \tag{14}$$

where $f = \inf_{b \in C_I}\{(\bar{X}-b)'\Sigma^{-1}(\bar{X}-b)\}$. LRK has the null distribution $\sum_{i=0}^{p} w(p,i,\Sigma)\chi_i^2$, where $w(p,i,\Sigma)$ is the weighting probability and χ_i^2 is the Chi-square random variable with i degrees of freedom (e.g., Shapiro (1985) for computation of the weighting probability). For the case of unknown Σ, we compare it with two well-known LR-based tests. One, by Perlman (1969), has the test statistic computed as

$$LRP = \frac{LPK_*}{n(d_* + 1)},$$

where d_* and LPK_* are computed as in (14) with $\hat{\Sigma}$ replacing Σ and the critical region being $LRP > c_\alpha^l$ with c_α^l determined by

$$0.5\,\Pr(\chi^2_{p-1}/\chi^2_{n-p+1} > c_\alpha^l) + 0.5\,\Pr(\chi^2_p/\chi^2_{n-p} > c_\alpha^l) = \alpha.$$

The other is the half-space test (H^+) proposed by Tang (1994) with the test statistic being LRP computed in any half-space that contains C_A and using the same critical region as in Perlman (1969). In this paper we follow the formula provided in Tang (1994) for computing the H^+ test statistic and the procedure provided in Larocque and Labarre (2004) for implementing their LLS tests.

All tests are performed at the 5% significance level. The number of replications is set to 10,000. The sample size is set to 20, 50, 100 and 500. For the LLS test we follow Theorem 2 of Larocque and Labarre (2004) to compute the critical regions as follows.

For $n = 20$,

$$T_{LL} \geq \begin{cases} 15 & 0 \leq m \leq 3 \\ 16 & 4 \leq m \leq 15 \\ 17 & 16 \leq m \leq 20, \end{cases}$$

where T_{LL} is their test statistic and m is the number of observations such that $x_{1i}x_{2i} < 0$. For $n = 50$,

$$T_{LL} \geq \begin{cases} 32 & 0 \leq m \leq 2 \\ 33 & 3 \leq m \leq 9 \\ 34 & 10 \leq m \leq 24 \\ 35 & 25 \leq m \leq 50. \end{cases}$$

Note that we encounter the computational problem of overflowing when we use their formulae for computing the critical region for a sample size even as large as 100, so we did not compare it with their test for $n = 100$, 500.

Tables 1-12 report the results of simulation studies of four bivariate distributions with two normal distributions with known and unknown Σ and two non-normal distributions for the case of $p = 2$ (the results of simulation studies for some examples of $p > 2$, which are not reported here, confirm our findings). The non-normal distributions were considered by Larocque and Labarre (2004). The first normal distribution has the covariance matrix (Case 1),

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \qquad (15)$$

with $\rho = 0.5$, while the second normal distribution (Case 2) has

$$\Sigma = \begin{pmatrix} 2 & 0.5 \\ 0.5 & 0.2 \end{pmatrix}. \qquad (16)$$

When Σ is assumed known, it can be shown that the EAMSSMP halfline test is identical to the FT test with $b_I = (1 \quad 1)'$ for the covariance structure in (15) with $-1 < \rho < 1$ (Lu (2011)), hence we do not report the FT test results in Tables 1 and 7.

Note that also for Σ in (15) with $-1 < \rho < 1$, Condition 2 is satisfied for any $b_I \geq 0$. Therefore, FT tests with any $b_I \geq 0$ are (asymptotically) unbiased and consistent. For Case 2 it can be shown that the halflines $l_{A(1 \quad 1)'}$ and $l_{A(\mu_1 \quad 0)'}$ form an obtuse angle for any $\mu_1 > 0$, hence the FT test is biased and inconsistent against the alternatives $(\mu_1 \quad 0)'$. The third distribution is a mixture of normal $0.2N(\mu, I) + 0.8N(\mu, 16I)$. The last is the Cauchy distribution, $Z_1^{-1}Y + \mu$, where Y is the centered bivariate normal with the covariance matrix of Case 1 above and Z_1 is a univariate independent standard normal variable. As discussed in Hahn and Klass (1980), Condition 3 is satisfied for these two non-normal distributions.

We first look at the estimation of sizes (Tables 1-6). Our results show that halfline tests, including EAMSSMP tests and FT tests, perform well; the estimated sizes of halfline tests converge to the nominal size quickly as n increases. Perlman's LR tests and LLS tests tend to underestimate it, while Tang's half-space tests do a better job than Perlman's LR tests. We now look at the estimation of powers. Our results confirm the biasedness and inconsistency of the FT test (Tables 9 and 10) for the normal distribution with Σ taking the structure of case 2 in Equation (16), although it can perform better than EAMSSMP tests for some alternatives, which is not surprising given the nature of SMP tests. In the case of normal distribution with known Σ (Tables 7 and 9), EAMSSMP tests perform better than LR tests when μ lies in the area around the halfline defined by b while the latter clearly has an advantage when μ lies on the boundary of the restricted parameter space defined by H_1. However, when Σ is unknown (Tables 8 and 10), EAMSSMP tests have some advantages over the LR-based tests and LLS tests. In particular, EAMSSMP tests always perform better when the alternative is close to H_0 and the sample size is small. Tables 5-12 show that EAMSSMP tests are less sensitive to the departure of the normality assumption than are LR- based tests.

EAMSSMP tests even perform better than the distribution-free LLS tests for the mixture normal distribution (Tables 5 and 11), but for the Cauchy distribution LLS tests appear to dominate when the alternative is not close to H_0 (Tables 6 and 12).

Table 1: Estimated sizes for the normal distribution of Case 1 when Σ is known.

n	T	LRK	LLS
20	0.050	0.051	0.027
50	0.051	0.048	0.036
100	0.052	0.053	×
500	0.054	0.051	×

Table 2: Estimated sizes for the normal distribution of Case 1 when Σ is unknown.

n	T	FT	LRP	H^+	LLS
20	0.059	0.068	0.035	0.047	0.026
50	0.055	0.059	0.036	0.049	0.037
100	0.053	0.055	0.039	0.051	×
500	0.049	0.049	0.037	0.051	×

Table 3: Estimated sizes for the normal distribution of Case 2 when Σ is known.

n	T	FT	LRK	LLS
20	0.051	0.052	0.048	0.030
50	0.054	0.048	0.050	0.038
100	0.051	0.048	0.052	×
500	0.047	0.047	0.047	×

Table 4: Estimated sizes for the normal distribution of Case 2 when Σ is unknown.

n	T	FT	LRP	H^+	LLS
20	0.059	0.067	0.035	0.042	0.028
50	0.059	0.060	0.041	0.048	0.036
100	0.050	0.054	0.042	0.050	×
500	0.048	0.054	0.043	0.052	×

Table 5: Estimated sizes for the mixture normal distribution when Σ is unknown.

n	T	FT	LRP	H^+	LLS
20	0.064	0.069	0.025	0.041	0.034
50	0.052	0.054	0.031	0.051	0.040
100	0.053	0.054	0.029	0.048	×
500	0.049	0.050	0.032	0.052	×

Table 6: Estimated sizes for the Cauchy distribution when Σ is unknown.

n	T	FT	LRP	H^+	LLS
20	0.058	0.061	0.010	0.018	0.034
50	0.055	0.053	0.009	0.019	0.035
100	0.052	0.048	0.011	0.021	×
500	0.051	0.048	0.010	0.022	×

Table 7: Estimated powers for the normal distribution of Case 1 when Σ is known.

n	T	LRK	LLS	T	LRK	LLS	T	LRK	LLS
	$\mu = (0.1 \quad 0)'$			$\mu = (0.2 \quad 0)'$			$\mu = (0.5 \quad 0)'$		
20	0.09	0.09	0.05	0.13	0.17	0.08	0.36	0.71	0.28
50	0.11	0.13	0.08	0.20	0.36	0.17	0.65	0.98	0.71
100	0.14	0.20	×	0.31	0.62	×	0.90	1	×
500	0.37	0.71	×	0.82	1	×	1	1	×
	$\mu = (0 \quad 0.1)'$			$\mu = (0 \quad 0.2)'$			$\mu = (0 \quad 0.5)'$		
20	0.08	0.09	0.04	0.14	0.18	0.07	0.37	0.71	0.30
50	0.10	0.13	0.08	0.21	0.37	0.17	0.66	0.98	0.72
100	0.14	0.21	×	0.32	0.61	×	0.89	1	×
500	0.36	0.71	×	0.83	1	×	1	1	×
	$\mu = (0.1 \quad 0.1)'$			$\mu = (0.2 \quad 0.2)'$			$\mu = (0.5 \quad 0.5)'$		
20	0.14	0.11	0.07	0.26	0.22	0.13	0.83	0.76	0.55
50	0.20	0.17	0.13	0.49	0.41	0.32	0.99	0.98	0.93
100	0.31	0.25	×	0.75	0.66	×	1	1	×
500	0.83	0.75	×	1	1	×	1	1	×

8 Conclusion

This paper studies halfline tests, including EAMSSMP and MSSMP tests. Our theoretical results offer some insights to halfline tests. We show that EAMSSMP, MSSMP and our proposed halfline tests are consistent and (asymptotically) unbiased, whereas some other halfline tests in the literature are (asymptotically) biased and inconsistent. The results of our simulation studies reveal that halfline tests can have better finite sample performance in some situations and are more robust against the normality assumption than are LR-based tests. Furthermore, the simplicity of implementation and general applicability of halfline tests illustrated in this paper provide an appealing alternative approach to the one-sided hypothesis testing problem.

APPENDIX: PROOFS

Proof of Lemma 4. Because $\delta_{k-1} \geq \delta_k$, l_{b_k} is rotating towards C_A as k decreases and $k = 2$ surely guarantees $b_k \in C_A$, it suggests that there always exists a b_k that satisfies the inequalities (11). We now show that such b_k is unique. Suppose that b_{k_*} that satisfies the inequalities (11) is found at k with a particular $\tilde{A}_k^* = \{\tilde{a}_{\tilde{j}}, \tilde{j} \in \mathbf{k}_*\}$. For any other $\tilde{A}_k\{\tilde{a}_{\tilde{j}}, \tilde{j} \in \mathbf{k}\}$, where $\mathbf{k}$ has the same dimension as $\mathbf{k}_*$, but $\mathbf{k} \neq \mathbf{k}_*$, the corresponding equiangular halfline l_{b_k} in $C_{\tilde{A}_k}$ always forms a larger angle with $l_{\tilde{a}_{\tilde{j}}}$ $\tilde{j} \in \mathbf{k}_*$, but $\tilde{j} \notin \mathbf{k}$; that is, $b_k'\tilde{a}_{\tilde{j}} < \delta_k$, $\tilde{j} \in \mathbf{k}_*$ and $\tilde{j} \in \mathbf{k}^c$. Hence, there cannot exist $b_k \in C_{\tilde{A}_k} \neq C_{\tilde{A}_k^*}$ such that the second inequality in (11) is satisfied while the first inequality is satisfied. At $k-1$, because $\delta_{k-1} \geq \delta_k$, it becomes more likely that $l_{b_{k-1}}$ forms an even larger angle with $l_{a_{\tilde{j}}}$, $\tilde{j} \in \mathbf{k}_*$, but $\tilde{j} \notin \mathbf{k}_-$, where $\mathbf{k}_- = \{\tilde{j} = 1, ..., k-1 : \tilde{j} \in \mathbf{p}\}$. Hence b_{k-1} that satisfies the inequalities (11) cannot occur if b_{k_*} exists. Finally, the second inequality in (11) says that l_{a_j}, $j \in \mathbf{k}_*^c$ forms a smaller

Table 8: Estimated powers for the normal distribution of Case 1 when Σ is unknown.

n	T	FT	LRP	H^+	LLS	T	FT	LRP	H^+	LLS	T	FT	LRP	H^+	LLS
	$\mu = (0.1 \quad 0)'$					$\mu = (0.2 \quad 0)'$					$\mu = (0.5 \quad 0)'$				
20	0.09	0.11	0.05	0.07	0.05	0.15	0.17	0.13	0.15	0.08	0.38	0.40	0.56	0.60	0.28
50	0.12	0.12	0.11	0.13	0.08	0.21	0.22	0.29	0.33	0.17	0.65	0.64	0.96	0.97	0.71
100	0.15	0.15	0.17	0.20	×	0.31	0.32	0.54	0.58	×	0.89	0.86	1	1	×
500	0.36	0.37	0.66	0.70	×	0.82	0.81	1	1	×	1	1	1	1	×
	$\mu = (0 \quad 0.1)'$					$\mu = (0 \quad 0.2)'$					$\mu = (0 \quad 0.5)'$				
20	0.10	0.11	0.05	0.07	0.05	0.15	0.16	0.12	0.14	0.07	0.38	0.40	0.57	0.61	0.29
50	0.12	0.12	0.10	0.12	0.08	0.22	0.22	0.29	0.33	0.17	0.67	0.65	0.96	0.97	0.71
100	0.14	0.15	0.17	0.20	×	0.32	0.32	0.54	0.58	×	0.89	0.87	1	1	×
500	0.37	0.37	0.66	0.70	×	0.82	0.82	1	1	×	1	1	1	1	×
	$\mu = (0.1 \quad 0.1)'$					$\mu = (0.2 \quad 0.2)'$					$\mu = (0.5 \quad 0.5)'$				
20	0.14	0.16	0.07	0.08	0.07	0.29	0.30	0.15	0.15	0.14	0.83	0.83	0.62	0.62	0.56
50	0.20	0.21	0.12	0.13	0.13	0.50	0.50	0.34	0.34	0.33	0.99	0.99	0.97	0.97	0.94
100	0.32	0.33	0.20	0.21	×	0.74	0.75	0.59	0.59	×	1	1	1	1	×
500	0.82	0.82	0.70	0.70	×	1	1	1	1	×	1	1	1	1	×

Table 9: Estimated powers for the normal distribution of Case 2 when Σ is known.

n	T	FT	LRK	LLS	T	FT	LRK	LLS	T	FT	LRK	LLS
	$\mu = (0.1 \quad 0)'$				$\mu = (0.2 \quad 0)'$				$\mu = (0.5 \quad 0)'$			
20	0.07	0.03	0.09	0.04	0.10	0.01	0.17	0.06	0.21	0	0.69	0.15
50	0.08	0.02	0.12	0.06	0.13 0	0.34	0.10	0.38	0	0.97	0.40	
100	0.11	0.01	0.20	$\times$	0.19	0	0.60	$\times$	0.59	0	1	$\times$
500	0.21	0	0.69	$\times$	0.51	0	1	$\times$	1	0	1	$\times$
	$\mu = (0 \quad 0.1)'$				$\mu = (0 \quad 0.2)'$				$\mu = (0 \quad 0.5)'$			
20	0.13	0.48	0.34	0.08	0.27	0.94	0.88	0.22	0.84	1	1	0.90
50	0.21	0.81	0.70	0.18	0.51	1	1	0.58	0.99	1	1	1
100	0.31	0.97	0.94	$\times$	0.76	1	1	$\times$	1	1	1	$\times$
500	0.84	1	1	$\times$	1	1	1	$\times$	1	1	1	$\times$
	$\mu = (0.15 \quad 0.05)'$				$\mu = (0.3 \quad 0.1)'$				$\mu = (0.4 \quad 0.1)'$			
20	0.13	0.09	0.11	0.07	0.27	0.16	0.20	0.14	0.33	0.09	0.27	0.18
50	0.20	0.13	0.15	0.13	0.50	0.26	0.39	0.32	0.60	0.13	0.52	0.40
100	0.31	0.17	0.24	$\times$	0.75	0.41	0.64	$\times$	0.84	0.17	0.79	$\times$
500	0.82	0.47	0.72	$\times$	1	0.94	1	$\times$	1	0.47	1	$\times$

Table 10: Estimated powers for the normal distribution of Case 2 when Σ is unknown.

n	T	FT	LRP	H^+	LLS	T	FT	LRP	H^+	LLS	T	FT	LRP	H^+	LLS
	$\mu = (0.1 \quad 0)'$					$\mu = (0.2 \quad 0)'$					$\mu = (0.5 \quad 0)'$				
20	0.08	0.04	0.06	0.07	0.04	0.11	0.02	0.12	0.14	0.06	0.23	0	0.56	0.61	0.15
50	0.08	0.02	0.10	0.12	0.06	0.14	0.01	0.28	0.32	0.10	0.38	0	0.96	0.97	0.39
100	0.11	0.01	0.17	0.20	×	0.19	0	0.55	0.59	×	0.59	0	1	1	×
500	0.21	0	0.66	0.69	×	0.52	0	1	1	×	0.99	0	1	1	×
	$\mu = (0 \quad 0.1)'$					$\mu = (0 \quad 0.2)'$					$\mu = (0 \quad 0.5)'$				
20	0.15	0.50	0.25	0.26	0.08	0.30	0.93	0.77	0.79	0.22	0.83	1	1	1	0.89
50	0.22	0.81	0.63	0.65	0.19	0.51	1	1	1	0.58	0.99	1	1	1	1
100	0.32	0.97	0.92	0.93	×	0.76	1	1	1	×	1	1	1	1	×
500	0.84	1	1	1	×	1	1	1	1	×	1	1	1	1	×
	$\mu = (0.15 \quad 0.05)'$					$\mu = (0.3 \quad 0.1)'$					$\mu = (0.4 \quad 0.1)'$				
20	0.14	0.12	0.07	0.07	0.07	0.29	0.20	0.16	0.16	0.15	0.34	0.12	0.20	0.20	0.18
50	0.21	0.14	0.13	0.14	0.13	0.50	0.28	0.33	0.33	0.32	0.60	0.14	0.46	0.46	0.41
100	0.31	0.18	0.20	0.20	×	0.74	0.42	0.59	0.59	×	0.85	0.19	0.77	0.77	×
500	0.82	0.48	0.70	0.70	×	1	0.93	1	1	×	1	0.48	1	1	×

Table 11: Estimated powers for the mixture normal distribution when Σ is unknown.

n	T	FT	LRP	H^+	LLS	T	FT	LRP	H^+	LLS	T	FT	LRP	H^+	LLS
			$\mu = (0.1 \quad 0)'$					$\mu = (0.2 \quad 0)'$					$\mu = (0.5 \quad 0)'$		
20	0.08	0.08	0.03	0.05	0.04	0.09	0.09	0.03	0.05	0.05	0.14	0.14	0.07	0.09	0.10
50	0.07	0.07	0.04	0.06	0.05	0.09	0.09	0.05	0.07	0.07	0.19	0.19	0.13	0.16	0.19
100	0.07	0.08	0.04	0.06	×	0.11	0.11	0.07	0.09	×	0.26	0.26	0.23	0.27	×
500	0.12	0.12	0.08	0.11	×	0.22	0.22	0.19	0.23	×	0.72	0.72	0.82	0.84	×
			$\mu = (0 \quad 0.1)'$					$\mu = (0 \quad 0.2)'$					$\mu = (0 \quad 0.5)'$		
20	0.07	0.08	0.03	0.05	0.04	0.08	0.09	0.03	0.05	0.05	0.14	0.15	0.07	0.09	0.10
50	0.07	0.07	0.04	0.06	0.05	0.09	0.09	0.05	0.07	0.07	0.18	0.19	0.12	0.16	0.17
100	0.07	0.07	0.04	0.06	×	0.11	0.11	0.07	0.09	×	0.26	0.26	0.23	0.27	×
500	0.11	0.11	0.08	0.10	×	0.23	0.23	0.20	0.24	×	0.71	0.71	0.82	0.85	×
			$\mu = (0.1 \quad 0.1)'$					$\mu = (0.2 \quad 0.2)'$					$\mu = (0.5 \quad 0.5)'$		
20	0.09	0.10	0.04	0.05	0.06	0.12	0.12	0.05	0.07	0.07	0.25	0.26	0.12	0.13	0.18
50	0.09	0.10	0.05	0.07	0.06	0.15	0.15	0.08	0.09	0.11	0.41	0.41	0.26	0.26	0.39
100	0.11	0.12	0.06	0.08	×	0.20	0.20	0.12	0.13	×	0.63	0.63	0.46	0.46	×
500	0.23	0.23	0.14	0.15	×	0.55	0.55	0.40	0.40	×	1	1	0.99	0.99	×

Table 12: Estimated powers for the Cauchy distribution when Σ is unknown.

n	T	FT	LRP	H^+	LLS	T	FT	LRP	H^+	LLS	T	FT	LRP	H^+	LLS
	$\mu = (0.1 \quad 0)'$					$\mu = (0.5 \quad 0)'$					$\mu = (5 \quad 0)'$				
20	0.08	0.08	0.01	0.02	0.05	0.13	0.15	0.05	0.06	0.20	0.73	0.67	0.71	0.73	1
50	0.06	0.06	0.02	0.03	0.07	0.12	0.13	0.05	0.07	0.45	0.71	0.65	0.72	0.74	1
100	0.06	0.06	0.02	0.02	$\times$	0.11	0.13	0.06	0.07	$\times$	0.72	0.66	0.74	0.75	$\times$
500	0.06	0.06	0.02	0.03	$\times$	0.12	0.13	0.07	0.08	$\times$	0.71	0.65	0.73	0.75	$\times$
	$\mu = (0 \quad 0.1)'$					$\mu = (0 \quad 0.5)'$					$\mu = (0 \quad 5)'$				
20	0.07	0.08	0.01	0.02	0.05	0.13	0.15	0.05	0.06	0.20	0.71	0.66	0.70	0.72	1
50	0.06	0.06	0.02	0.02	0.06	0.11	0.12	0.05	0.07	0.45	0.72	0.65	0.72	0.74	1
100	0.06	0.06	0.02	0.02	$\times$	0.12	0.13	0.06	0.07	$\times$	0.71	0.65	0.73	0.75	$\times$
500	0.06	0.05	0.02	0.03	$\times$	0.11	0.12	0.07	0.08	$\times$	0.73	0.67	0.74	0.76	$\times$
	$\mu = (0.1 \quad 0.1)'$					$\mu = (0.5 \quad 0.5)'$					$\mu = (5 \quad 5)'$				
20	0.08	0.09	0.02	0.02	0.07	0.21	0.24	0.09	0.10	0.36	0.94	0.96	0.92	0.92	1
50	0.08	0.08	0.02	0.03	0.10	0.21	0.23	0.10	0.11	0.74	0.94	0.95	0.93	0.93	1
100	0.08	0.07	0.02	0.03	$\times$	0.20	0.23	0.12	0.13	$\times$	0.94	0.95	0.93	0.94	$\times$
500	0.07	0.07	0.02	0.03	$\times$	0.21	0.23	0.12	0.13	$\times$	0.94	0.96	0.94	0.94	$\times$

angle with $l_{b_{k_*}}$ than $l_{a_{\tilde{j}}}$, $\tilde{j} \in \mathbf{k}_*$ does, so $l_{a_j} \subset C_{\tilde{A}^*_k}$, $\forall j$. Then the equiangular halfline for $\mathbf{k}_+ = \{\tilde{j} = 1, ..., k+1 : \tilde{j} \in \mathbf{p}\}$, which satisfies the second inequality in (11) for $\mathbf{k}^c_+$, must lie outside $C_{\tilde{A}}$, and so cannot satisfy the first inequality; as a result, b_{k+1}, which satisfies the two inequalities (11) at the same time, cannot occur if b_{k_*} exists. ■

Proof of Lemma 6. The proof can be better shown geometrically. In the polyhedral cone C_A, the first three constraints mean that $l_b \subset C_A$ and $0 < \varphi(l_b, l_{a_j}) < \pi/2$, $\forall j$. When $p > 2$, suppose that an affine hyperplane cut through the cone, then the intersection of the hyperplane and the cone is a convex polygon with p sides. Denote O as the point that is the intersection of the halfline, l_b, and the hyperplane. Denote d_j, $j = 1, ..., p$, the point of the intersection of the edge, l_{a_j}, and the hyperplane. The optimization problem in (13) is then equivalent to the maximization of the sum of the distance between O and d_j, $\forall j$. Given a convex polygon with p sides, there is a unique solution for the optimization of the sum of the distances $\sum_{j=1}^{p} Od_j$. To prove this, denote $d_{j,j+1}$, where $j+1$ is reset to $j+1-p$ if $j+1 > 3$, the point that is the intersection of the line $Od_{j,j+1}$ perpendicular to the line d_jd_{j+1}. Let α_j be the angle at the point d_j and α_{j1} be its right side sub-angle. Then

$$\sum_{j=1}^{p} Od_j = 2\sum_{j=1}^{p}[\frac{d_jd_{j,j+1}}{\cos\alpha_{j1}} + \frac{d_{j-1}d_j - d_{j-1}d_{j-1,j}}{\cos(\alpha_j - \alpha_{j1})}], \tag{A.1}$$

where $j-1$ is reset to $j-1+p$ if $j-1 < 1$. It can be shown that the first order conditions with respect to $d_jd_{j,j+1}$ and α_{j1}, $\forall j$, result in a unique solution. That means that if the third constraint is not binding, there is a unique solution of l_b with $\varphi(l_b, l_{a_j}) > \arccos\delta_l$, $\forall j \in \mathbf{p}$. If the third constraint is binding, then there exists the unique corner solution. Note that if any $\alpha_j > \pi/2$, Equation (A.1) can be easily modified, but the solution to its first order condition is still unique. ■

Proof of Theorem 1. $\Sigma^{-1}b_I > 0$ is equivalent to $A'b > 0$. Therefore, from Equation (5) we have $\cos\varphi(l_b, l_{a_j}) > 0$, $\forall j$. From Equation (4) for any $\eta \in C_A$ or $\mu \in C_I$ we have the power $\beta(\varphi(\mu)) > \alpha$ for a given n and $\lim_{n\to\infty}\beta(\varphi(\mu)) = 1$. This proves the sufficiency.

For the necessity, suppose there exists an unbiased and consistent halfline test with one element of $\Sigma^{-1}b_I$ being non-positive. Then the corresponding $\cos\varphi(l_b, l_{a_j}) \leq 0$. Hence, for all η (or μ) lying on the corresponding edge of C_A (or C_I), from Equation (4) we have $\beta(\varphi(\mu)) \leq \alpha$ for a given n and $\lim_{n\to\infty}\beta(\varphi(\mu)) = 0$. This contradicts the assumption of unbiasedness and consistency. This proves the necessity. ■

Proof of Corollary 1. For EAMSSMP tests, because $C_A \subset H_A^+$, the equiangular halfline in C_A means $0 < \varphi(l_b, l_{a_j}) = \varphi_c < \pi/2$, $j \in \mathbf{p}$, where $\varphi_c = \arccos\delta$ and $\delta = (s'\Sigma s)^{-1/2}$, hence $\cos\varphi(l_b, l_{a_j}) > 0$, $\forall j$, which is equivalent to the satisfaction of Condition 2. For MSSMP tests implemented by the procedure discussed in Section 4, it is obvious that $0 < \varphi(l_b, l_{a_j}) < \varphi_{c,k} < \pi/2$, $\forall j$, where $\varphi_{c,k} = \arccos\delta_k$. For our proposed halfline tests, the optimization stated in (13) ensures the solution b such that Condition 2 is guaranteed. Therefore, the unbiasedness and consistency results follow from Equation (4). ■

Proof of Corollary 2. It can be easily seen that, with the replacement of Σ with Q, halfline test statistics in Equations (2), (3) and (7) become $\sigma^{-1}T$ and $\sigma^{-1}T_x$. Replacing unknown σ with its unbiased estimate $\hat{\sigma}$ in the test statistics results in the null $t_{n'}$ distribution. The unbiasedness and consistency properties follow immediately from Equation (4) with the replacement of $\Phi(\cdot)$ with the cumulative distribution function of the student t distribution with n' degrees of freedom. ■

Proof of Theorem 2. For any $l_{b_I} \subset C_I$, there always exists a Σ such that l_{b_I} forms an obtuse angle with at least one of the edges of C_A. Let us first consider the case $p = 2$.

Suppose Σ takes the form in Equation (8) and $b_I = (1 \quad 1)'$, then

$$\Sigma^{-1} b_I = (\sigma_1^2 \sigma_2^2 - \sigma_{12}^2)^{-1} \begin{pmatrix} \sigma_2^2 - \sigma_{12} \\ \sigma_1^2 - \sigma_{12} \end{pmatrix}.$$

Because $\sigma_1 > 0$, $\sigma_2 > 0$ and $-1 < \sigma_{12}/(\sigma_1 \sigma_2) < 1$, $\sigma_1^2 \sigma_2^2 - \sigma_{12}^2 > 0$, there always exist σ_1, σ_2 and σ_{12} such that either $\sigma_2^2 - \sigma_{12}$ or $\sigma_1^2 - \sigma_{12}$ is negative (Σ in Equation (16) is such an example). It can be easily seen that this remains true for any fixed $b_I \geq 0$. Geometrically, for any fixed $b_I \neq (1 \quad 1)'$, the halfline l_{Ab_I} is located on either side of the halfline $l_{A(1 \quad 1)'}$ in C_A. Therefore, it is always possible to find Σ such that l_{Ab_I} forms an obtuse angle with one of the edges of C_A. Hence the biasedness and inconsistency results follow. We now prove the results in the general case of $p \geq 2$. We shall show that there always exists a Σ such that the angle between l_{Ab_I} and l_{a_j} for at least one $j \in \mathbf{p}$ is greater than $\pi/2$, hence prove the results. Because one can always rearrange the order of μ such that such j appears in the first element of μ, it suffices to show that there always exists a Σ such that the angle between l_{Ab_I} and l_{a_1} is greater than $\pi/2$. Suppose that $\varphi(l_{Ab_I}, l_{a_1}) < \pi/2$ for a fixed b_I. Then, there exists a new Σ such that $\varphi(l_{Ab_I}, l_{a_1}) > \pi/2$ by changing the sign of the element in the $(1, 1)$ position of the A matrix. Then, for all $\eta \in C_1 \subset C_A$, where $C_1 = \{l_\eta : \varphi(l_\eta, l_{Ab_I}) \geq \pi/2\}$, we have $\beta(\varphi(\eta)) \leq \alpha$ for a given n and $\lim_{n\to\infty} \beta(\varphi(\eta)) = 0$. ■

Proof of Lemma 7. By the multivariate normality result of Sepanski (1996),

$$\sqrt{n}\hat{\Sigma}^{-\frac{1}{2}}(\bar{X} - \mu) \xrightarrow{d} N(0, I),$$

as $n \to \infty$. Therefore, $0 < \lim_{n\to\infty} \varphi(l_b, l_{a_j} | \hat{\Sigma}) < \pi/2$, $\forall j$, where b is the one in EAMSSMP, MSSMP or our proposed halfline tests. From Equation (4), it can be seen that $\lim_{n\to\infty} \beta(\varphi(\mu)|\mu = 0, \hat{\Sigma}) = \alpha$ and $\lim_{n\to\infty} \beta(\varphi(\mu)|\mu \in C_I, \hat{\Sigma}) = 1$. ■

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