MASA Workshop

Water-cycle Management (Phil Howlett and Julia Piantadosi)

1. Problem description

We wish to construct a simulation model for water cycle management in a suburban development. We suppose that all stormwater flows into a capture sam and is subsequently transferred to a holding dam which will supply the water to residents and local industry for non-potable usage.
We have developed an SVG (Scalar Vector Graphics) simulation model for a system of two connected storage dams and for a prototype Mawson Lakes model. SVG is a language for describing two-dimensional graphics. In our case this enables us to depict each component of the water cycle management system. The animation is controlled by ECMAScript functions that operate the various objects. To run the SVG simulations it is necessary to download Adobe SVG Viewer if you do not already have this on your computer. A free download of Adobe SVG Viewer is available from the following website:

http://www.adobe.com/svg/viewer/install

Installing Adobe SVG Viewer

1. Double-click the download installer.
2. Follow the on-screen instructions.
3. If you are not using Internet Explorer for Windows, then you will need to restart your browser before viewing SVG.

Once this software has been installed you will be able to run SVG simulation files.

2. **Example: A SVG simulation of two connected dams**

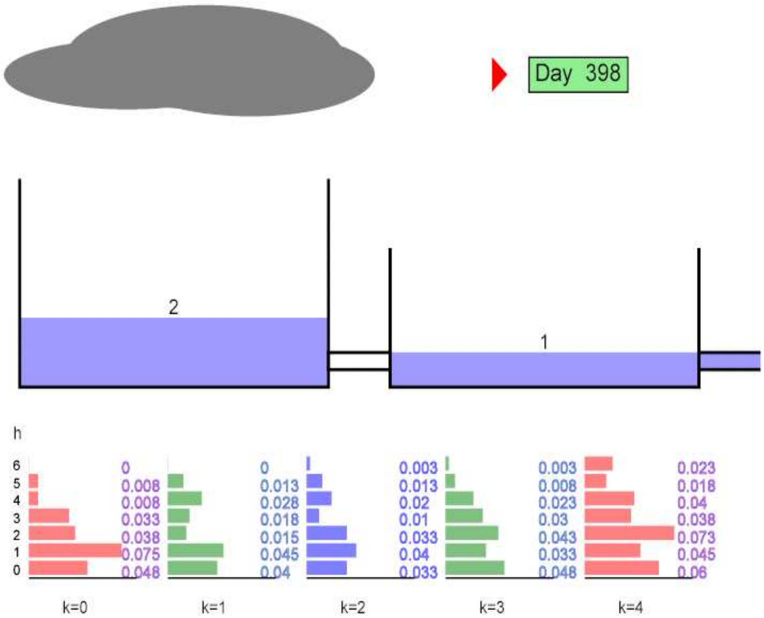

Figure 1: Simulation of the system of two connected dams.

The first dam is the capture dam and the second is the holding dam. The capacity of the capture dam is 6 units and the capacity of the holding dam is 4 units. Stormwater from synthetic rainfall generated randomly from a known probability distribution fills the capture dam. Water is pumped from the capture dam to the holding dam in accordance with a simple control policy. Water is drawn from the holding dam at a constant daily rate of one unit per day. The rainfall probabilities have been chosen so that the average input for the model is 1 unit. The simulation shows the level of water in each dam and shows the water being pumped from the capture and holding dams, the total input and the total overflow from the system. The set of histograms represent the relative frequencies of occurrence for the different states of the system. E.g. $z_1 = 0$ and $z_2 = 0$ is when the capture and holding dams are empty, $z_1 = 2$ and $z_2 = 3$ is when the capture dam contains two units and holding dam contains three units etc. In the long run the relative frequencies approximate the steady state probabilities. The steady state probabilities are the theoretical proportion of time spent in each state.

3. **Student Activities**

- We can use beakers of water to demonstrate how various water management policies cope with random rainfall. Try the following simple policies:

  1. always empty the first dam
  2. pump to fill the second dam if possible

  Which is the policy that wastes the least amount of water?

- Develop a mathematical model of a simple system using stochastic matrices.

Example:

We will consider a simple example to show the structure of the transition probability matrix. The state (z_1, z_2) of the system is defined by $Z_1 = \{0, 1, 2, 3\}$, $Z_2 = \{0, 1, 2\}$. Let us consider the management policy in which we pump m units from the first dam whenever possible. For this example we will take the control parameter $m = 2$. Thus we pump two units of water from the first dam to the second dam if the first dam contains two or more units of water and we pump one unit from the second dam unless the dam is empty. We define p_r to be the probability that r units enter the first dam and $p_s{}^+ = \sum_{r=s}^{\infty} p_r$. We refer to some particular transitions described in table 2 for this example.

Table 1: Typical Transitions for a simple example

initial state	new state	units pumped from dam 1	units pumped from dam 2	probability
$(0,0)$	$(0,0)$	0	0	p_0
	$(1,0)$	0	0	p_1
	$(2,0)$	0	0	p_2
	$(3,0)$	0	0	p_3^+
$(1,0)$	$(1,0)$	0	0	p_0
	$(2,0)$	0	0	p_1
	$(3,0)$	0	0	p_2^+
$(2,0)$	$(0,2)$	2	0	p_0
	$(1,2)$	2	0	p_1
	$(2,2)$	2	0	p_2
	$(3,2)$	2	0	p_3^+
$(3,0)$	$(1,2)$	2	0	p_0
	$(2,2)$	2	0	p_1
	$(3,2)$	2	0	p_2^+

Table 2: Typical Transitions for a simple example cont.

initial state	new state	units pumped from dam 1	units pumped from dam 2	probability
$(0,1)$	$(0,0)$	0	1	p_0
	$(1,0)$	0	1	p_1
	$(2,0)$	0	1	p_2
	$(3,0)$	0	1	p_3^+
$(1,1)$	$(1,0)$	0	1	p_0
	$(2,0)$	0	1	p_1
	$(3,0)$	0	1	p_2^+
$(2,1)$	$(0,2)$	2	1	p_0
	$(1,2)$	2	1	p_1
	$(2,2)$	2	1	p_2
	$(3,2)$	2	1	p_3^+
$(3,1)$	$(1,2)$	2	1	p_0
	$(2,2)$	2	1	p_1
	$(3,2)$	2	1	p_2^+
$(0,2)$	$(0,1)$	0	1	p_0
	$(1,1)$	0	1	p_1
	$(2,1)$	0	1	p_2
	$(3,1)$	0	1	p_3^+
$(1,2)$	$(1,1)$	0	1	p_0
	$(2,2)$	0	1	p_1
	$(3,2)$	0	1	p_2^+
$(2,2)$	$(0,2)$	2	1	p_0
	$(1,2)$	2	1	p_1
	$(2,2)$	2	1	p_2
	$(3,2)$	2	1	p_3^+
$(3,2)$	$(1,2)$	2	1	p_0
	$(2,2)$	2	1	p_1
	$(3,2)$	2	1	p_2^+

In this example we order the states (z_1, z_2) as follows

$$(0,0)\ \ (1,0)\ \ (2,0)\ \ (3,0)\ \ (0,1)\ \ (1,1)\ \ (2,1)\ \ (3,1)\ \ (0,2)\ \ (1,2)\ \ (2,2)\ \ (3,2).$$

Therefore we can construct the transition matrix as

$$A = \begin{bmatrix} p_0 & 0 & 0 & 0 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_1 & p_0 & 0 & 0 & p_1 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_2 & p_1 & 0 & 0 & p_2 & p_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_3^+ & p_2^+ & 0 & 0 & p_3^+ & p_2^+ & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_1 & p_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_2 & p_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_3^+ & p_2^+ & 0 & 0 \\ 0 & 0 & p_0 & 0 & 0 & 0 & p_0 & 0 & 0 & 0 & p_0 & 0 \\ 0 & 0 & p_1 & p_0 & 0 & 0 & p_1 & p_0 & 0 & 0 & p_1 & p_0 \\ 0 & 0 & p_2 & p_1 & 0 & 0 & p_2 & p_1 & 0 & 0 & p_2 & p_1 \\ 0 & 0 & p_3^+ & p_2^+ & 0 & 0 & p_3^+ & p_2^+ & 0 & 0 & p_3^+ & p_2^+ \end{bmatrix}.$$

We note that the zeros represent transitions that are not allowed by the management policy.
We solve the equation

$$p = Ap \qquad (1)$$

to determine the steady state vector.

Scalable Vector Graphics (SVG) is a language for describing two-dimensional graphics. SVG allows for three types of graphic objects: vector graphic shapes (e.g., paths consisting of straight lines and curves), images and text. In the case of the dam system, this enables the user to construct each component of the water management system. The objects are created in a viewport which is a rectangular region onto which graphics elements can be drawn. There is a coordinate system, which defines locations and distances on the current canvas. The basic graphic elements are 'path', 'rect', 'circle', 'ellipse', 'line', 'polyline' and 'polygon'. These are standard shapes which are predefined in SVG as a convenience for common graphical operations. There are two parts to the script. The first part defines the objects that appear on the screen. The second part defines the functions that implement the simulation and modify the parameters of the the on-screen objects.

For the system of two connected dams, the dam and controls are drawn in a coordinate system that is 400 units wide, 300 high, and has the upper left corner at $(0, 0)$. When the SVG script is first loaded, the initialisation function draws and labels the initial dam levels. The dam is drawn as a polyline and the water as a rectangle. The dam components are defined relative to the origin and the group is then translated to an appropriate position on the page. During the simulation the water height and location are altered by a function which also changes the position and text of the level indication. The animation is controlled by ECMAScript functions that operate on the various document objects. The function rain() generates a quantity of rain to flow into the first dam, then increments the dam levels.

- Develop a simple simulation model using SVG.

Stochastic Matrices and Markov Chains

1. A typical example

We all know that weather is difficult to predict in advance. The Bureau of Meteorology uses enormous computers to predict likely pressure patterns and air movements for the next seven days. Such predictions are never accurate enough to say with certainty that it will or will not rain at a particular place on a particular day. For long-term predictions we might well decide that the best we can do is to look at past rainfall records and estimate the probability of rain on a particular day.
Let us suppose our observations suggest the following probabilities. If it is dry today the probability it will be dry tomorrow is 0.9 and the probability it will be wet is 0.1. If it is wet today the probability it will be dry tomorrow is 0.7 and the probability it will be wet is 0.3. Our model uses two possible states D and W. We write

$$P[\ D \to D\ ] = 0.9,\ \ P[\ D \to W\ ] = 0.1,\ \ P[\ W \to D\ ] = 0.7 \text{ and } P[\ W \to W\ ] = 0.3.$$

Note the general rules

$$P[\ D \to D\ ] + P[\ D \to W\ ] = 0.9 + 0.1 = 1.$$

and

$$P[\ W \to D\ ] + P[\ W \to W\ ] = 0.7 + 0.3 = 1.$$

We can represent the observed conditional probabilities as a matrix array with two rows and two columns. The first column represents transitions out of state D and the second column represents transitions out of state W. The first row represents transitions into state D and the second row represents transitions into state W. The transition matrix is given by

$$A = \begin{bmatrix} 0.9 & 0.7 \\ 0.1 & 0.3 \end{bmatrix}.$$

The current state of the system can be described by a two dimensional vector. The vector

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

means the system is in state D while the vector

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

means the system is in state W. If the system starts in state D on day t we write

$$x(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and we work out the probable state on day $t+1$ by the matrix multiplication

$$x(t+1) = Ax(t) = \begin{bmatrix} 0.9 & 0.7 \\ 0.1 & 0.3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.9 \\ 0.1 \end{bmatrix}$$

which means there is probability 0.9 that tomorrow (day $t+1$) will be dry and probability 0.1 that tomorrow will be wet. For the next day we have

$$x(t+2) = Ax(t+1) = \begin{bmatrix} 0.9 & 0.7 \\ 0.1 & 0.3 \end{bmatrix} \begin{bmatrix} 0.9 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 0.88 \\ 0.12 \end{bmatrix}$$

and for the third day

$$x(t+3) = Ax(t+2) = \begin{bmatrix} 0.9 & 0.7 \\ 0.1 & 0.3 \end{bmatrix} \begin{bmatrix} 0.88 \\ 0.12 \end{bmatrix} = \begin{bmatrix} 0.876 \\ 0.124 \end{bmatrix}.$$

By looking at the individual calculations you will see that

$$P[\text{ day } t+1 \text{ is dry }] = P[\ D \to D\ ] \times P[\text{ day } t \text{ is dry }] + P[\ W \to D\ ] \times P[\text{ day } t \text{ is wet }]$$

and

$$P[\text{ day } t+1 \text{ is wet }] = P[\ D \to W\ ] \times P[\text{ day } t \text{ is dry }] + P[\ W \to W\ ] \times P[\text{ day } t \text{ is wet }].$$

In symbolic form these equations are written as

$$x_1(t+1) = (0.9)x_1(t) + (0.7)x_2(t)$$

and

$$x_2(t+1) = (0.1)x_1(t) + (0.3)x_2(t).$$

By looking at the numbers in the probability distributions $x(t), x(t+1), x(t+2), \ldots$ we might guess that the system is heading towards some sort of long-term equilibrium. For this to happen we would expect that there might be a steady state probability vector

$$p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

such that

$$p = Ap.$$

This means that

$$\begin{aligned} p_1 &= 0.9p_1 + 0.7p_2 \\ p_2 &= 0.1p_1 + 0.3p_2. \end{aligned}$$

Since p is a probability vector we must also have $p_1 \geq 0$, $p_2 \geq 0$ and $p_1 + p_2 = 1$. There is a certain amount of redundant information (we have only two unknowns and there are three equations) but we can nevertheless solve this system of equations to find a steady state probability vector

$$p = \begin{bmatrix} 0.875 \\ 0.125 \end{bmatrix}.$$

Since $x(t+3) \approx p$ our calculations apparently confirm our hypothesis that the steady state probability p is the long term limiting probability distribution. Hence, in the long term, we might expect that seven of every eight days will be dry and one will be wet. Of course we should remember that our analysis started from a dry day. What happens if we start from a wet day?

For each initial probability distribution

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

the infinite chain of successive probability distributions $x(t), x(t+1), x(t+2), \ldots, x(t+s), \ldots$ generated by the formulae

$$\begin{aligned} x(t+1) &= Ax(t) \\ x(t+2) &= Ax(t+1) \\ x(t+3) &= Ax(t+2) \\ &\vdots \\ x(t+s) &= Ax(t+s-1) \\ &\vdots \end{aligned}$$

is called a Markov Chain.
Suppose we wish to do two steps at once to generate the sequence $x(t), x(t+2), x(t+4), \ldots$. Show that this sequence is also a Markov Chain with transition matrix A^2. Calculate the matrix A^2 and verify that it is indeed a transition matrix.
From a practical view you might argue that the transition probabilities in the above example should really be different for different times in the year. We would certainly expect that the probability of a wet day is higher in winter than it is in summer. To describe this more complicated behaviour the transition matrix should depend on time and we would need a sequence of transition matrices $A(1), A(2), \ldots, A(365)$ to describe the model. The Markov Chain would now be determined by the formula

$$\begin{aligned} x(t+1) &= A(t)x(t) \\ x(t+2) &= A(t+1)x(t+1) \\ x(t+3) &= A(t+2)x(t+2) \\ &\vdots \\ x(t+s) &= A(t+s-1)x(t+s-1) \\ &\vdots \end{aligned}$$

and we would no longer expect the distribution to converge to a steady state limit. What would you expect to happen? You might consider what happens if you try to do 365 steps at once to generate a sequence $x(t), x(t+365), x(t+730), \ldots$. Is this sequence a Markov Chain? Can you calculate a transition matrix for this sequence? What about the sequence $x(t+1), x(t+366), x(t+731), \ldots$? Is this a Markov Chain too?

2. An example with an absorbing state

In the previous example it was possible to switch between each of the two states in one step. The next example is a Markov Chain where some transitions are not possible and where one state eventually absorbs all the probability. There are three states and the transition matrix is given by

$$A = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.25 & 0.5 & 0 \\ 0.25 & 0.5 & 1 \end{bmatrix}.$$

The matrix A is said to be lower triangular and we can see that this means the transitions $2 \to 1$, $3 \to 1$ and $3 \to 2$ are not possible in one step. Show that the matrix A^2 is also lower triangular and hence deduce that even if we make two steps the transitions $2 \to 1$, $3 \to 1$ and $3 \to 2$ are still not possible. Since we can move into state 3 from each of the other states but we can never move out of state 3 we might expect state 3 to eventually capture all the probability. If we begin in state 1 with

$$x(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

then we have

$$x(t+1) = Ax(t) = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.25 & 0.5 & 0 \\ 0.25 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.25 \\ 0.25 \end{bmatrix},$$

$$x(t+2) = Ax(t+1) = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.25 & 0.5 & 0 \\ 0.25 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.25 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.5 \end{bmatrix}$$

and

$$x(t+3) = Ax(t+2) = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.25 & 0.5 & 0 \\ 0.25 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 0.25 \\ 0.25 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.125 \\ 0.1875 \\ 0.6875 \end{bmatrix}$$

from which we might guess that the limiting distribution is

$$p = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

What equation would you solve to confirm that p is the invariant (steady) state vector for this Markov Chain? What does this limit mean?

3. An example with two separate cycles

In this example there are two separate cycles in the Markov Chain. You will see from the transition matrix that the transitions $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$ are possible but other transitions are not. We have four states with

$$A = \begin{bmatrix} 0.9 & 0.7 & 0 & 0 \\ 0.1 & 0.3 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 1 \end{bmatrix}.$$

What happens eventually if we start in state 1 or state 2? What happens if we start in states 3 or 4? Explain why this system behaves like two separate Markov Chains.

4. An example where some transitions are delayed

Consider the transition matrix

$$A = \begin{bmatrix} 0.5 & 0 & 1 \\ 0.5 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

There are three states. Note that the transitions $2 \to 1$, $2 \to 2$, $3 \to 2$, $1 \to 3$ and $3 \to 3$ are not possible in one step. We calculate

$$A^2 = \begin{bmatrix} 0.25 & 1 & 0.5 \\ 0.25 & 0 & 0.5 \\ 0.5 & 0 & 0 \end{bmatrix},$$

$$A^3 = \begin{bmatrix} 0.625 & 0.5 & 0.25 \\ 0.125 & 0.5 & 0.25 \\ 0.25 & 0 & 0.5 \end{bmatrix}$$

and

$$A^4 = \begin{bmatrix} 0.5625 & 0.25 & 0.625 \\ 0.3125 & 0.25 & 0.125 \\ 0.125 & 0.5 & 0.25 \end{bmatrix}.$$

The matrix A^4 has all positive elements and we write $A^4 > 0$. This means that every transition is possible in four steps.

5. An introduction to the general theory of Markov Chains

Definition: If a Markov Chain has k possible states, labelled $1, 2, \cdots, k$, then the probability that the system will move to state i at time $t+1$ given that it was in state j at time t is denoted by a_{ij} where $0 < a_{ij} < 1$ and is called the **transition probability** from state j to state i. The matrix A, with entries a_{ij}, is called the **transition matrix** for the Markov Chain. The probabilities in each column sum to 1. Thus $a_{1j} + a_{2j} + \cdots + a_{kj} = 1$ for each $j = 1, 2, \ldots, k$.

Definition: The **state vector** is a column vector $x = x(t)$ whose i^{th} component is the probability that the system is in state i at time t. The probabilities in the state vector sum to one. Thus $x_1(t) + x_2(t) + \cdots + x_k(t) = 1$.

Theorem: If A is the transition matrix of a Markov Chain and $x(t)$ is the state vector at time t then

$$x(t+1) = Ax(t).$$

Therefore

$$\begin{aligned} x(t+1) &= Ax(t) \\ x(t+2) &= Ax(t+1) \\ x(t+3) &= Ax(t+2) \\ \vdots \quad &\vdots \quad \vdots \\ x(t+s) &= Ax(t+s-1) \end{aligned}$$

and so, once the initial state vector $x(t)$ and the transition matrix are known, the state vector at $x(t+s)$ at any future time $t+s$ can be found.

Theorem: Let A be the transition matrix for a Markov Chain. Then A^s is the transition matrix for a Markov Chain describing the s-stage evolution given by

$$x(t+s) = A^s x(t).$$

Theorem: Let $A = [a_{ij}]$ be the transition matrix for a Markov Chain and suppose $a_{ij} > 0$ for all (i, j). In such cases for convenience we write $A > 0$. Then there exists a unique steady state probability vector p such that $p = Ap$. More generally if the matrix $A^s > 0$ then there exists a unique steady state vector.

References

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