Year 11 Mathematics Awareness Workshop – Recurrence relations and difference equations

Robot walking

Imagine you have a robot that can take steps of 1metre or 2metres in length and we want to work out how many ways this robot can walk n metres.

Length of walk	Walks	Total number of walks	Number of walks starting with 1	Number of walks starting with 2
1	1	1	1	0
2	2 1+1	2	1	1
3	2+1 1+1+1 1+2	3	2	1
Ask students to suggest walks for 4metres				
4	2+1+1 2+2 1+1+1+1 1+1+2 1+2+1	5	3	2

Each walk obviously starts with either a 1m step or a 2m step.
For the 3m walk there are 2 walks that start with 1 and 1 walk that starts with 2.
For the 4m walk there are 3 walks that start with 1 and 2 walks that start with 2.

Does any one recognise the numbers in the total column?

Fibonacci sequence

If no one says Fibonacci, get them to look for patterns (students who don't recognise the Fibonacci sequence may notice that each number in the total column is the sum of the two previous numbers, but it's more likely that students without knowledge of sequences will notice that the diagonals across the three columns are the same)

We can write a formula for the number of ways a robot can walk n metres as follows

$$w_n = w_{n-1} + w_{n-2}, \qquad w_1 = 1 \qquad w_2 = 2.$$

The number of ways for the robot to walk n metres is the sum of the number of ways it can walk (n-1) metres (because if we add a 1 to the front, these are then ways to walk n metres) and the number of ways it can walk (n-2) metres (adding a 2 gives ways to walk n metres).

The Fibonacci sequence is an example of order 2 recursion, as each number depends on the previous two. A difference equation is simply an equation defining a recursive sequence. We use difference equations to model situations or quantities that change over time, for example populations or amounts of money. The equation for compound interest, for example $a_n = 1.05a_{n-1}$ for 5% interest, (explain a_n represents amount of money after n months, years etc) is an example of order 1 recursion. As a difference equation we would write this as $a_n – a_{n-1} = 0.05a_n$.

The problem with defining something recursively is that if we wanted to calculate (for example) how many ways the robot could walk 100 metres we would have to work out 100 terms of a sequence which would be very time consuming. This is why we want to solve recurrence relations to find a general term of the sequence.

Sequence of averages

If we define a sequence where each number is the average of the previous two numbers, so in mathematical notation

$$s_n = \frac{1}{2}(s_{n-1} + s_{n-2})$$

what do you think will happen as we calculate more and more terms of the sequence?

This sequence is a convergent sequence, which means as we calculate more and more terms of the sequence the terms will get closer and closer to (or converge to) a particular value. If we choose $s_0 = 2$ and $s_1 = 8$, what do you think it will converge to?

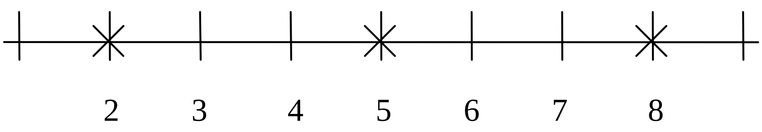
Expect someone will guess 5

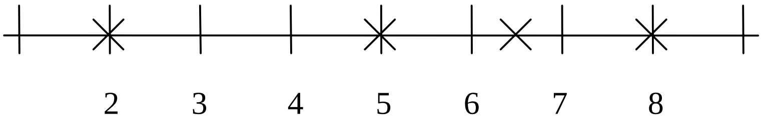
Choose two positive integers, any numbers you like but make it easier for yourself by choosing two numbers less than 10, call these s_0 and s_1 and calculate the next four terms of the sequence, up to s_5. Please don't choose the same numbers as the person next to you, or 2 and 8 – I'll work out the sequence for those numbers as an example.

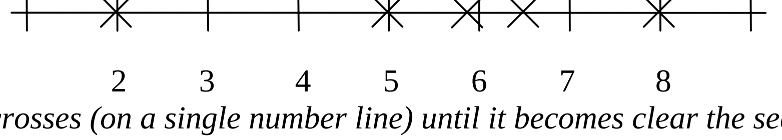
While students are doing this draw the following table on the board, then ask several students for the values they choose/calculated to fill in about 10 lines or so of the table, some example lines are shown below:

s_0	s_1	s_2	s_3	s_4	s_5
2	8	5	6.5	5.75	6.125
3	6	4.5	5.25	4.875	5.0625

It doesn't look like the sequence beginning with 2 and 8 is converging to 5 – it actually converges to 6. We can explain this by looking at a number line. Each time we take the average value the resulting number is in the middle of the two previous numbers.







Keep drawing crosses (on a single number line) until it becomes clear the sequence does converge to 6.

We can solve the recurrence relation to see exactly what the sequence converges to. We can prove that it is 6 for the case when we start with 2 and 8 and see what happens for any other starting values. *Get students to guess what their sequence would converge to.*

Recurrence relations are often solved by trial and error. We make an educated guess about what the solution might be and see if our guess satisfies the equation.

Because I know what the answer is, I will try as a solution

$s_n = r^n$,

where r is some number. Solutions to recurrence relations often look like this, if you remember what happens when you solve a compound interest recurrence relation, the solution for the 5% example is $a_n = (1.05)^n a_0$, so this is a reasonable guess.

If we substitute this into the equation, we have

$r^n = ½ (r^{n-1} + r^{n-2})$

which we can rearrange to give

$r^n – ½ (r^{n-1} + r^{n-2}) = 0.$

If we take r^{n-2} out as a common factor, we have

$r^{n-2} (r^2 – ½ r – ½) = 0$

If r was equal to zero, then every number in the sequence would be zero and obviously that is not the case so the other part of our equation must equal zero. This is just a quadratic and we all know how to factorise a quadratic. When we do so we find

$(r – 1)(r + ½) = 0$

so

$r = 1$ or $r = – ½$.

This gives us two possible solutions

$s_n = 1^n = 1$ or $s_n = (–½)^n$

The first solution is just a constant, and this happens because if we choose our two starting numbers to be the same, the sequence would be the same number repeated over and over. The second solution is more interesting. In general the solution will be a combination of these two solutions, that is

$s_n = a + b (–½)^n$

where a and b are constants that can be found by using the initial or starting values.

To find the solution for particular starting values, s_0 and s_1 we look at the general formula when n = 0 and n = 1. This gives us two equations

$s_0 = a + b$

$s_1 = a – ½ b$

that we can solve to find a and b in terms of s_0 and s_1.

We find *(by solving simultaneous eq'ns – demonstrate this)* that

$a = 1/3 (s_0 + 2s_1)$

$b = 2/3 (s_0 – s_1)$

So for our example with $s_0 = 2$ and $s_1 = 8$ we find

$a = 1/3(2 + 16) = 6$

$b = 2/3(2 – 8) = – 4$

so the general formula is

$s_n = 6 – 4 (- ½)^n$

(If time permits – check for the first few terms, otherwise skip this bit)

Checking for the first few values we find

$s_0 = 6 – 4 = 2$

$s_1 = 6 – 4 (–½) = 6 + 2 = 8$

$s_3 = 6 – 4 (–½)^2 = 6 – 4 (1/4) = 6 – 1 = 5$

$s_4 = 6 – 4 (–½)^3 = 6 – 4 (–1/8) = 6 + ½ = 6.5$

$s_5 = 6 – 4 (–½)^4 = 6 – 4 (1/16) = 6 – ¼ = 5.75$

The value we are adding to or subtracting from six gets smaller and smaller each time – so you can see that eventually the sequence will converge to 6.

Looking at the general formula

$s_n = a – b (- ½)^n$

as n gets larger 2^n becomes larger and larger, so $1/2^n$ will get closer and closer to zero and the sequence converges to a. In our example, a is 6.

Get the students to calculate

$a = 1/3 (s_0 + 2s_1)$

for their own sequence. Does it match with what they thought it would converge to?

Check with the students whose sequences were written on the board what their sequences converged to.